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# Basic Representations of Extended Affine Lie Algebras

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1. As is well known, an affine Lie algebra  $\underline{g}(A)$  has two kinds of natural realizations in terms of an underlying finite-dimensional simple Lie algebra. I would like to start my talk with a short sketch of them following [7]. Let  $A = (a_{ij})_{i,j=0,\dots,\ell}$  be a generalized Cartan matrix of type  $X_N^{(k)}$ , and  $\mathfrak{g}$  the complex finite-dimensional simple Lie algebra of type  $X_N$ , where  $X$  is  $A, B, C, \dots$ , or  $G$ . Let  $\mu$  be the automorphism of  $\mathfrak{g}$ , which permutes the Chevalley generators of  $\mathfrak{g}$  according to an automorphism of order  $k$  of its Dynkin diagram. Set  $\varepsilon = \exp(2\pi i/k)$  and denote by  $\mathfrak{g}_j(\mu)$  the eigenspace of  $\mu$  with eigenvalue  $\varepsilon^j$ . Then 
$$\sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{g}_j(\mu) + \mathbb{C}c + \mathbb{C}d$$
 is an affine Lie algebra

associated to  $A$ , where  $c$  is a generator of the center and

$$d = t \frac{d}{dt}.$$

The another realization is constructed in the following way. Let  $\sigma$  be a Coxeter transformation on  $\mathfrak{g}$  which commutes

with  $\mu$ , and  $h$  be the order of  $\sigma$ . Then each eigenvalue of  $\sigma\mu$  has the form  $\omega^j$ , where  $\omega = \exp(2\pi i/kh)$ . Set

$$\mathfrak{p}_j(\mu, \sigma) = \{x \in \mathfrak{p} ; \sigma\mu x = \omega^j x\}$$

for  $j \in \mathbb{Z}$ . Then one has a so-called "principal realization"

$$\underline{g}(A) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{p}_j(\mu, \sigma) + \mathbb{C}c + \mathbb{C}d_0.$$

Note that under these two realizations  $d$  does not correspond to  $d_0$ , because  $\langle d, \alpha_i \rangle = \delta_{0,i}$  and  $\langle d_0, \alpha_i \rangle = 1$  for every simple root  $\alpha_i$ .

2. Associated to either of these two realizations, an affine Lie algebra  $\underline{g}(A)$  admits an extension

$$\widetilde{\underline{g}}_J'(A) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{p}_j(\mu) + \mathbb{C}c + \sum_{n \in k\mathbb{Z}} \mathbb{C}d'_n$$

or 
$$\widetilde{\underline{g}}_J(A) = \sum_{j \in \mathbb{Z}} t^j \otimes \mathfrak{p}_j(\mu, \sigma) + \mathbb{C}c + \sum_{n \in \mathbb{Z}} \mathbb{C}d_n,$$

where  $d'_n = t^{n+1} \frac{d}{dt}$  and  $d_n = t^{nkh+1} \frac{d}{dt}$ . Functions  $J'$

and  $J$  define the Lie brackets

$$[d'_m, d'_n] = (n-m) d'_{m+n} + J'(m) \delta_{m,-n} c,$$

$$[d_m, d_n] = \underbrace{(n-m)}_{kh} d_{m+n} + J(m) \delta_{m,-n} c.$$

It is known that if  $J$  (resp.  $J'$ ) is not trivial the sub-

algebra  $\mathcal{H} = \sum_{n \in \mathbb{Z}} \mathbb{C}d_n + \mathbb{C}c$  (resp.  $\mathcal{H}' = \sum_{n \in k\mathbb{Z}} \mathbb{C}d'_n + \mathbb{C}c$ ) is

isomorphic to the Virasoro algebra.

In this talk, we shall call  $\tilde{\underline{g}}'_{J'}(A)$  (resp.  $\tilde{\underline{g}}_J(A)$ ) the extension of the 1st (resp. 2nd) type.

Note that there is a remarkable difference between these two extensions. Denote by  $\underline{\overset{\circ}{g}}$  the finite-dimensional subalgebra of  $\underline{g}(A)$  generated by  $\{e_i, f_i; i=1, \dots, \ell\}$ . Then in the 1st extension  $\underline{\overset{\circ}{g}}$  commutes with  $\mathcal{V}'$ , while in the 2nd extension the centralizer  $Z_{\tilde{\underline{g}}_J(A)}(\mathcal{V})$  of  $\mathcal{V}$  in  $\tilde{\underline{g}}_J(A)$  coincides with  $\mathbb{C}c + \underline{\overset{\circ}{h}}$ , where  $\underline{\overset{\circ}{h}} = \underline{\overset{\circ}{g}} \cap \underline{h}$  is a Cartan subalgebra of  $\underline{\overset{\circ}{g}}$ .

The 1st type extension may be said to be the picture of G. Segal [10] and I. B. Frenkel [2][3]. If  $k = 1$  and  $J'(m) = J'_0(m) = (m^3 - m)/12$ , then  $\mathcal{V}'$  is just the standard form of the Virasoro algebra. We set  $\tilde{\underline{g}}'(A) = \tilde{\underline{g}}'_{J'_0}(A)$ . The basic representation of an extended affine Lie algebra  $\tilde{\underline{g}}'(A)$  was first discovered by G. Segal [10] in case of  $A = A_1^{(1)}$ , and the theory has been developed by I. B. Frenkel in case of  $k = 1$  and  $J' = J'_0$ .

It seems to me that there exists no isomorphism between  $\tilde{\underline{g}}'_{J'}(A)$  and  $\tilde{\underline{g}}_J(A)$ .

Now I want to consider representations of the 2nd type extensions. My conjecture is that "for any dominant integral form  $\Lambda \in P_+$ , there will exist a unique central extension  $J$ , such that the action of  $\underline{g}(A)$  on  $L(\Lambda)$  can be extended to that of  $\tilde{\underline{g}}_J(A)$ ."

At present, I can verify this conjecture only for a few special cases:  $\Lambda = \Lambda_0$  and  $A = A_1^{(1)}$ ,  $A_2^{(1)}$  or  $A_2^{(2)}$ .

3. In case of the basic representation of  $A_1^{(1)}$ , the whole story is most simple and most beautiful. So, from now on, let me restrict my talk on  $A = A_1^{(1)}$  and  $\Lambda = \Lambda_0$ . It is well known that the basic representation  $L(\Lambda_0)$  of  $A_1^{(1)}$  is constructed on the space  $\mathbb{C}[x_1, x_3, x_5, \dots]$  of polynomial functions in  $x_i$ 's with odd indices. We can prove that  $\tilde{g}_J(A_1^{(1)})$  acts on  $L(\Lambda_0)$  if and only if  $J(m) = m(2m^2+1)/6$ , and that the action is given by

$$\pi : -d_n \longmapsto L_{2n} = \frac{1}{2} \sum_{\substack{j \in \mathbb{Z} \\ j=\text{odd}}} : a_j a_{2n-j} : ,$$

where  $a_j = \frac{\partial}{\partial x_j}$  and  $a_{-j} = jx_j$  for a positive odd integer  $j$ , and  $:$  denotes the normal product. These operators  $\{L_{2n}\}_{n \in \mathbb{Z}}$  are called the Virasoro operators.

Now look at the weights diagram of  $L(\Lambda_0)$ . It is known that the set  $P(\Lambda_0)$  of all weights of  $L(\Lambda_0)$  is given by

$$P(\Lambda_0) = \left\{ \Lambda_0 + q\delta + p\alpha_1 ; \begin{array}{l} p, q \in \mathbb{Z} \\ q \leq -p^2 \end{array} \right\} ,$$

and the set of all maximal weights is given by

$$\text{Max}(\Lambda_0) = W \cdot \Lambda_0 = \left\{ \Lambda_0 + q\delta + p\alpha_1 ; \begin{array}{l} p, q \in \mathbb{Z} \\ q = -p^2 \end{array} \right\} ,$$

where  $W$  is the Weyl group and  $\delta = \alpha_0 + \alpha_1$  is the fundamental imaginary root.

The multiplicity of each weight can be calculated by the Weyl-Kac character formula, and one has

$$\text{Mult}_{\Lambda_0}(\lambda - n\delta) = p(n) \quad ,$$

where  $\lambda$  is a maximal weight and  $p(n)$  is the partition number, i.e.,

$$\mathcal{G}(x)^{-1} = \prod_{j=1}^{\infty} (1-x^j)^{-1} = \sum_{n=0}^{\infty} p(n)x^n \quad .$$

Now we are interested in the problem to write down functions in each weight space explicitly. It is easily seen that a function in a weight space  $L(\Lambda_0)_{\Lambda_0+q\delta+p\alpha_1}$  has the degree  $-(2q+p)$ , where the degree of a function is counted with the rule that the degree of  $x_j$  is equal to  $j$ . Consider the action of the Virasoro algebra  $\mathcal{V}$ . From  $[d_0, d_n] = 2nd_n$ , one sees that the operator  $L_{2n}$  maps a weight space  $L(\Lambda_0)_{\lambda}$  to  $L(\Lambda_0)_{\lambda+n\delta}$ ;

$$L_{2n} : L(\Lambda_0)_{\lambda} \longrightarrow L(\Lambda_0)_{\lambda+n\delta} \quad .$$

So  $\sum_{n \in \mathbb{Z}} L(\Lambda_0)_{\lambda+n\delta}$  is stable under the action of  $\mathcal{V}$ , and

by counting the multiplicity, we see that  $\sum_{n \in \mathbb{Z}} L(\Lambda_0)_{\lambda+n\delta}$

is an irreducible  $\mathcal{V}$ -module. So each maximal weight vector  $f$  is described as a solution of the system of linear differential equations

$$L_{2n} f = 0 \quad \text{for } n \geq 1 \quad ,$$

which are equivalent to

$$L_2 f = 0 \quad \text{and} \quad L_4 f = 0 \quad .$$

It is known from the Sato's theory that maximal weight vectors cover the homogeneous polynomial solutions of the KdV-hierarchies. So we obtain the following theorem:

Theorem. A homogeneous polynomial  $f$  in  $\mathbb{C}[x_1, x_3, x_5, \dots]$  is a solution of the KdV-hierarchies if and only if  $L_2 f = 0$  and  $L_4 f = 0$ , where

$$L_2 = \frac{1}{2} \left( \frac{\partial}{\partial x_1} \right)^2 + \sum_{\substack{j=1 \\ j=\text{odd}}}^{\infty} j x_j \frac{\partial}{\partial x_{j+2}}$$

$$L_4 = \frac{\partial^2}{\partial x_1 \partial x_3} + \sum_{\substack{j=1 \\ j=\text{odd}}}^{\infty} j x_j \frac{\partial}{\partial x_{j+4}} .$$

Owing to the  $\mathcal{H}$ -irreducibility of  $\sum_{n \in \mathbb{Z}} L(\Lambda_0)_{\lambda+n\delta}$ ,

all vectors in other weight spaces are obtained by iterated operations of  $L_{2n}$ 's ( $n < 0$ ) to those maximal weight vectors.

4. I want to point out here that in the Segal's picture or in the Frenkel's picture,  $\sum_{n \in \mathbb{Z}} L(\Lambda_0)_{\lambda+n\delta}$  can never

be expected to be irreducible under the action of  $\mathcal{H}'$ , because  $\mathcal{H}'$  commutes with  $\underline{\hat{g}}$ . Let me explain about this more precisely. For a maximal weight  $\lambda$ , we set

$$V_\lambda = \sum_{n=0}^{\infty} L(\Lambda_0)_{\lambda-n\delta}.$$

Next decompose  $L(\Lambda_0)$  under the action of  $\underline{\mathfrak{g}} = \underline{\mathfrak{sl}}(2, \mathbb{C})$ , and for a positive odd integer  $n$  let  $W_n$  denote the sum of all  $n$ -dimensional irreducible  $\underline{\mathfrak{g}}$ -submodules of  $L(\Lambda_0)$ .

Then it is easily seen from the weights diagram that for every  $\lambda \in \text{Max}(\Lambda_0)$  there exists a positive integer  $n_\lambda$  satisfying  $V_\lambda \cap W_n \neq \{0\}$  for any odd integer  $n$  larger than  $n_\lambda$ . As to the decomposition  $V_\lambda = \sum_{n \geq 1} (V_\lambda \cap W_n)$

of  $V_\lambda$ , each  $V_\lambda \cap W_n$  is stable under the action of  $\mathcal{H}'$ , because of  $[\mathcal{H}', \underline{\mathfrak{g}}] = 0$ . Thus we have proved that in the extension of the 1st type, each  $V_\lambda$  is not  $\mathcal{H}'$ -irreducible and so maximal weight vectors cannot be characterized as the highest weight vectors with respect to  $\mathcal{H}'$ .

5. The representation theory of extended affine Lie algebras suggests or induces some problems. Consider the group orbit through the highest weight vector  $1$  in the completion of  $L(\Lambda_0)$ . It is well known that  $G(A) \cdot 1$  coincides with the set of  $\mathcal{C}$ -functions of the KdV-equation. What is the differential equation which characterizes  $G(A)$ -orbit through  $1$ ? Or, take a subgroup  $G' \subset \tilde{G}(A)$ . What is the differential equation characterizing  $G'$ -orbit through  $1$ ? Now I want to point out an interesting fact; take a subalgebra  $\underline{\mathfrak{g}}' = \underline{\mathfrak{s}} + \mathcal{H}$ , where  $\underline{\mathfrak{s}}$  is the principal subalgebra, then we have



$$f \in G' \cdot 1 \implies f \otimes f \in \underline{g}'\text{-highest component in } L(\Lambda_0) \otimes L(\Lambda_0)$$

$$\iff \langle u, f \otimes f \rangle = 0 \quad \text{for every } u \text{ in lower } \underline{g}'\text{-components}$$

$$\iff D(f \circ f) = 0 \quad \text{for } \forall D = \text{Hirota's bilinear differential operator in (non-reduced) BKP-hierarchies.}$$

Thus the  $G'$ -orbit is related to the non-reduced BKP-hierarchies.

According to the theory of Sato, Kashiwara, Miwa, Jimbo and Date, solutions of the BKP-equations are described by the orbit of  $O(\infty)$ . I cannot tell why BKP-hierarchies appear within the framework of  $A_1^{(1)}$ . I want to conclude this section by noticing a fact which may be related to the above phenomenon. Consider the natural inclusions  $A_1^{(1)} \subset D_4^{(2)}$  and  $D_n^{(2)} \subset D_{2n}^{(2)}$  for  $n \in \mathbb{N}$ . Then the basic representation of  $D_4^{(2)}$  is irreducible when restricted to the subalgebra  $A_1^{(1)}$ , and it is nothing but the basic representation of  $A_1^{(1)}$ ; i.e.,

$$L(\Lambda_0; A_1^{(1)}) = L(\Lambda_0; D_4^{(2)}) .$$

In a similar way, one has

$$L(\Lambda_0; D_n^{(2)}) = L(\Lambda_0; D_{2n}^{(2)}) .$$

So, under the sequence of inclusions

$$A_1^{(1)} \subset D_4^{(2)} \subset D_8^{(2)} \subset \dots \subset D_{2^n}^{(2)} \subset \dots \subset D_{2^\infty}^{(2)},$$

one has

$$\begin{aligned} L(\Lambda_0; A_1^{(1)}) &= L(\Lambda_0; D_4^{(2)}) = L(\Lambda_0; D_8^{(2)}) \\ &= \dots = L(\Lambda_0; D_{2^\infty}^{(2)}). \end{aligned}$$

6. In this section, we take  $V = \mathbb{C}[x_1, x_2, x_3, \dots]$  and set

$$a_k = \frac{\partial}{\partial x_k}, \quad a_{-k} = kx_k \quad (\text{for } k \in \mathbb{N}),$$

$$a_0 = 0$$

$$\text{and } L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} : a_k a_{n-k} : \quad (\text{for } n \in \mathbb{N}).$$

We denote by  $\mathcal{H}$  the Lie algebra spanned by  $\{1, a_k, L_k; k \in \mathbb{Z}\}$ , and consider the representation of  $\mathcal{H}$  on  $V$ . Then we can prove that

$$f \in H \cdot 1 \Rightarrow f \otimes f \in \text{the highest component of } V \otimes V$$

$$\Leftrightarrow \langle u, f \otimes f \rangle = 0 \quad \text{for every } u \text{ in lower components}$$

$$\Leftrightarrow D(f \circ f) = 0 \quad \text{for } \nabla_D = \text{Hirota's bilinear differential operator in KP-hierarchies.}$$

These facts suggest that the Virasoro algebra is deeply related to hierarchies. Further discussions as to the relation between the Virasoro algebra and modified KP-

hierarchies will be shown in the joint work [15] with H. Yamada.

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